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Density of polynomials in some L^2 spaces on radial rays in the complex plane

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ABSTRACT

In this paper, we study the necessary and sufficient conditions for the density of the linear space of matrix polynomials in a linear space of square integrable functions with respect to a matrix of measures supported on a set of radial rays of the complex plane. The connection with a completely indeterminate Hamburger matrix moment problem is stated. Vector valued functions associated in a natural way with a function defined in the union of the radial rays are used. Thus, our first aim is the construction of a linear space of square integrable functions with respect to a matrix of measures supported on a set of radial rays and a positive semi-definite matrix acting on the discrete part of the corresponding inner product. An isometric transformation which allows to reduce the problem of density to the case of the real line is introduced. Finally, some examples of such spaces are shown and its completeness is studied in detail.

1. Introduction

In this paper, we will analyze some linear spaces of complex-valued functions which are defined on radial rays of the complex plane

$$L_N = \{\lambda \in \mathbb{C} : \ Im(\lambda^N) = 0\}, \tag{1}$$

where $N \in \mathbb{N}$. L_N can be represented as a union either of $2N$ radial rays or N lines of the complex plane

$$L_N = \bigcup_{k=0}^{2N-1} \{x\widehat{\varepsilon}^k, \ x \geqslant 0\} = \bigcup_{k=0}^{N-1} \{x\widehat{\varepsilon}^k, \ x \in \mathbb{R}\}, \tag{2}$$

where $\widehat{\varepsilon} = \cos\frac{\pi}{N} + i\sin\frac{\pi}{N}$ is a root of unity of order $2N$.

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We introduce the following notation:

$$L'_N = L_N \backslash \{0\}, \quad L'_{N,k} = \{x\hat{\varepsilon}^k,\ x > 0\}, \quad k = 0, 1, \ldots, 2N-1, \tag{3}$$

and, thus

$$L'_N = \bigcup_{k=0}^{2N-1} L'_{N,k}. \tag{4}$$

The algebra of all $n \times n$ matrices with complex entries will be denoted by $\mathbb{C}_{n\times n}$. $\mathbb{C}^{\geqslant}_{n\times n}$ will be the cone of all positive semi-definite matrices in the above algebra. We denote by $\mathfrak{B}(L'_N)$ the σ-algebra of Borel subsets of L'_N. A $\mathbb{C}^{\geqslant}_{N\times N}$-valued measure $\mathbf{M}$ on $\mathfrak{B}(L'_N)$ is a σ-additive function $\mathbf{M}(\Delta) = (m_{i,j}(\Delta))_{i,j=0}^{N-1}$ $(\Delta \in \mathfrak{B}(L'_N))$ from $\mathfrak{B}(L'_N)$ into $\mathbb{C}^{\geqslant}_{N\times N}$ (see, e.g. [15] for the general definition of a positive semi-definite matrix measure).

We will denote by $\mathbf{M}'_\tau = (m'_{\tau;i,j})_{i,j=0}^{N-1}$ the Radon–Nikodym derivative of $\mathbf{M}$ with respect to the trace measure $\tau_\mathbf{M} = \sum_{i=0}^{N-1} m_{i,i}$. As usual, $L^1_{\tau_\mathbf{M}}(L'_N)$ will denote the linear space of $\mathbb{C}$-valued $\mathfrak{B}(L'_N)$-measurable functions on L'_N which are integrable with respect to $\tau_\mathbf{M}$.

Let f be an arbitrary $\mathbb{C}$-valued function on L_N. We denote

$$\vec{f}_s = \vec{f}_s(\lambda) = (f(\lambda), f(\lambda\varepsilon), \ldots, f(\lambda\varepsilon^{N-1})),$$

where $\varepsilon = \cos\frac{2\pi}{N} + i\sin\frac{2\pi}{N}$ is a root of unity of order N and $\lambda \in L_N$.

A $\mathbb{C}$-valued function f on L_N *has derivative of order* m, $m \geqslant 1$, at zero if the derivatives of order k at zero for the function f restricted to the line $L_{n,j} = \{x\hat{\varepsilon}^j,\ x \in \mathbb{R}\}$, exist for all $0 \leqslant j \leqslant N-1$, $1 \leqslant k \leqslant m$, and the derivatives of the same order coincide. As the value of the derivative of order m of f at zero, we choose the common value of the derivatives of order m. We will use the standard notation $f^{(m)}(0)$ for the derivative of order m at zero.

Let $A = (a_{i,j})_{i,j=0}^{N-1} \in \mathbb{C}^{\geqslant}_{N\times N}$ be an arbitrary matrix. If A is not the zero matrix, then we define

$$\mathfrak{r} = \mathfrak{r}(A) = \max\{i:\ 0 \leqslant i \leqslant N-1,\ a_{i,i} \neq 0\} \tag{5}$$

Taking into account that $|a_{i,j}|^2 \leqslant a_{i,i}a_{j,j}$, then all elements $a_{i,j}$ such that $\max(i,j) > \mathfrak{r}$ vanish.

Set

$$J_\lambda = (g_{k,l}(\lambda))_{k,l=0}^{N-1}, \quad g_{k,l}(\lambda) = \frac{1}{N\varepsilon^{kl}\lambda^l}, \quad \lambda \in L'_N. \tag{6}$$

In the sequel, we will deal with the linear space of functions given below.

Definition 1.1. Let $A \in \mathbb{C}^{\geqslant}_{N\times N}$, $N \in \mathbb{N}$, and $\mathbf{M}$ be a bounded $\mathbb{C}^{\geqslant}_{N\times N}$-valued measure on $\mathfrak{B}(L'_N)$. Let us consider a set $L^2_{\mathbf{M},A}(L_N)$ of $\mathbb{C}$-valued functions on L_N defined as follows:
$f \in L^2_{\mathbf{M},A}(L_N)$ if and only if

(i) The restriction of f to L'_N is a $\mathfrak{B}(L'_N)$-measurable function.
(ii) $\vec{f}_s(\lambda) J_\lambda \mathbf{M}'_\tau J^*_\lambda \vec{f}^*_s(\lambda) \in L^1_{\tau_\mathbf{M}}(L'_N)$.
(iii) If A is not the zero matrix then f has the derivative of order $\mathfrak{r} = \mathfrak{r}(A)$ at zero.

Given a function $f \in L^2_{\mathbf{M},A}(L'_N)$, we denote

$$\vec{f}_d = \left(f(0), f'(0), \ldots, f^{(N-1)}(0)\right) \tag{7}$$

and

$$\vec{f}_t = \left(f(0), \frac{f'(0)}{1!}, \ldots, \frac{f^{(N-1)}(0)}{(N-1)!} \right). \tag{8}$$

By convention, when the derivatives do not exist one should assume that the corresponding entry is 0.

We define an inner product in $L^2_{\mathbf{M},A}(L'_N)$ as follows

$$(f,g)_{\mathbf{M},A} = \int_{L'_N} \vec{f}_s(\lambda) J_\lambda \mathbf{M}'_\tau(\lambda) J^*_\lambda \vec{g}^*_s(\lambda) d\tau_{\mathbf{M}} + \vec{f}_d A \vec{g}^*_d, \quad f,g \in L^2_{\mathbf{M},A}. \tag{9}$$

Notice that from condition (ii) and the Cauchy–Schwarz inequality we get convergence of the integral. As usual, for $f \in L^2_{\mathbf{M},A}$ the norm associated with the inner product is

$$\|f\|_{\mathbf{M},A} = \sqrt{(f,f)_{\mathbf{M},A}}. \tag{10}$$

We will consider equivalence classes of such functions with respect to $(\cdot,\cdot)_{\mathbf{M},A}$. Then $L^2_{\mathbf{M},A}$ becomes a unitary space with inner product $(\cdot,\cdot)_{\mathbf{M},A}$. We shall prove that this space is complete if and only if either $A = 0$ or $\mathfrak{r}(A) = 0$. Below we will show that in some particular cases the space $L^2_{\mathbf{M},A}$ becomes the standard linear space of square integrable functions with respect to a positive measure supported either on L_N or on a discrete Sobolev space. To study the density of complex polynomials in $L^2_{\mathbf{M},A}$ we shall construct an isometric transformation which maps $L^2_{\mathbf{M},A}$ onto a dense subset of an L^2 space of vector-valued functions on $\mathbb{R}$ with respect to a $\mathbb{C}^{\geqslant}_{N\times N}$-valued measure on $\mathbb{R}$.

Let $\mathbf{M}$ be an arbitrary $\mathbb{C}^{\geqslant}_{N\times N}$-valued measure on $\mathfrak{B}(\mathbb{R})$, the σ-algebra of Borel subsets of $\mathbb{R}$, with finite matrix moments

$$S_k = \int_{\mathbb{R}} x^k d\mathbf{M}(x), \quad k \in \mathbb{Z}_+. \tag{11}$$

The set of all $\mathbb{C}^{\geqslant}_{N\times N}$-valued measures on $\mathfrak{B}(\mathbb{R})$ with moments $\{S_k\}_{k=0}^\infty$ is denoted by $\mathcal{V} = \mathcal{V}(\{S_k\}_{k=0}^\infty)$. The description of the matrix measures $\mathcal{V}$ with associated moments $\{S_k\}_{k=0}^\infty$ is known in the literature as the matrix Hamburger moment problem [10, p. 52].

Let $L^2_{\mathbf{M}} = L^2_{\mathbf{M}}(\mathbb{R})$ be the linear space of $\mathbb{C}_{N\times N}$-valued functions on $\mathbb{R}$ which are square integrable with respect to $\mathbf{M}$. Denote by $\mathbb{P}_{N\times N}$ the set of all $N\times N$ matrices whose entries are complex polynomials. We denote $\mathbb{P} = \mathbb{P}_{1\times 1}$. We assume that for every

$$P(x) = I_N x^k + A_{k-1}x^{k-1} + A_{k-2}x^{k-2} + \cdots + A_0 \in \mathbb{P}_{N\times N},$$

where $A_i \in \mathbb{C}_{N\times N}$, $i = 0, 1, \ldots, k-1$, $k \in \mathbb{Z}_+$, and I_N is the unit matrix of order N, the following condition holds

$$\int_{\mathbb{R}} P(x) d\mathbf{M}(x) P^*(x) \text{ is a nonsingular matrix.} \tag{12}$$

Applying the Gram–Schmidt pseudo-orthogonalization method (see [1, pp. 577–578]) to the sequence $\{x^n I_N\}_{n=0}^\infty$ we obtain a sequence of orthonormal matrix polynomials $\{P_k(x)\}_{k=0}^\infty$ such that $P_k(x)$ is a matrix polynomial of degree k. Also,

$$\int_{\mathbb{R}} P_k(x) d\mathbf{M}(x) P^*_l(x) = I_N \delta_{k,l}, \quad k,l \in \mathbb{Z}_+. \tag{13}$$

We can associate a $N \times N$ block tridiagonal Jacobi matrix J to these polynomials. The corresponding linear operator $\mathbf{J}$ in $l^2 = \{(x_l)_{l \in \mathbb{Z}_+} : x_l \in \mathbb{C}, \sum_{k=0}^{\infty} |x_k|^2 < \infty\}$ is symmetric. Let (m_-, m_+) be its deficiency index. Notice that $N = max\{m_-, m_+\}$. The matrix Hamburger moment problem is said to be *completely indeterminate* if $m_+ = m_- = N$. In this case $\mathbb{P}_{N\times N}$ will be dense in L^2_M if and only if (see [12, Theorem 1.1, p. 249])

$$\int_{\mathbb{R}} \frac{d\mathbf{M}(x)}{x-\lambda} = -\{C^*(\lambda)[I+U] - iA^*(\lambda)[I-U]\}\{D^*(\lambda)[I+U] \\ -iB^*(\lambda)[I-U]\}^{-1}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{14}$$

where U is a constant unitary matrix and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are holomorphic matrix functions which can be computed explicitly from the sequence of moments $\{S_k\}_{k=0}^{\infty}$. We shall use this result to obtain conditions for the density of polynomials in $L^2_{\mathbf{M},A}$. In [11] an extension to the matrix case of the classical Riesz's theorem is stated. Indeed, for the set of solutions of the matrix moment problem the author proves that the linear space of matrix polynomials is dense in the corresponding weighted L^2 space for those matrices of measures such that their Stieltjes transform is an extremal point, in the sense of convexity, of the image set. In [4] the author study N-extremal matrices of measures of an indeterminate matrix moment problem and their connection with density of polynomials. For the case $N = 2$ the density of polynomials in some analogous spaces was studied in [7].

Matrix of measures appear in a natural way in the framework of higher order recurrence relations satisfied by polynomial sequences. In [2] the author deals with the measure of orthogonality associated with inner products such that the multiplication operator by x^N is symmetric. Then, the corresponding sequences of orthonormal polynomials satisfy a $2N + 1$ recurrence relation. Furthermore, as a particular case of these inner products, a characterization of Sobolev type inner products with mass points at the origin for the terms involving derivatives is given. Notice that if one considers a general Sobolev inner product, then the existence of a polynomial h such that the multiplication by h is a symmetric operator with respect to the inner product is equivalent to the fact that the measures involving the derivatives are discrete and their supports are related to the zeros of h, as it was proved in [5]. The connection between these Sobolev type orthogonal polynomials and matrix orthogonal polynomials was studied in [3].

On the other hand, in [13] the authors deal with Hermitian inner products with respect to scalar measures supported on harmonic algebraic curves $L = \{\lambda \in \mathbb{C} : Imh(\lambda) = 0\}$ where h is a polynomial of degree N with complex coefficients. The multiplication operator by h is an Hermitian operator and the corresponding sequence of orthogonal polynomials satisfies a $2N + 1$ recurrence relation which leads to a matrix three term recurrence relation for a sequence of matrix polynomials orthogonal with respect to a matrix of measures supported on the real line. The study of such recurrence relations and their basic solutions as difference equations has been done in [14].

Notice than in a recent contribution [8] the notion of convergence in measure $\mathbf{M}$ has been introduced. This generalizes the notion of convergence in measure with respect to a scalar measure and takes into account the matrix structure of $\mathbf{M}$. There, given a subset $\mathcal{S}$ of square matrices the characterization of the closure of sets of $\mathcal{S}$-valued measurable functions under convergence in measure is presented.

The structure of the paper is as follows. In Section 2 we analyze the density of polynomials in the linear space $L^2_{\mathbf{M},A}(L_N)$. The key idea is the construction of an isometric transformation of the above linear space onto another linear space of vector valued functions which are square integrable with respect to a new matrix of measures $\widetilde{\mathbf{M}}$. A characterization of the density of these polynomials in terms of the Hilbert transform of the matrix measure $\widetilde{\mathbf{M}}$ is presented. In Section 3 we characterize the completeness of the linear space $L^2_{\mathbf{M},A}(L_N)$ in terms of the matrix A. Finally, in Section 4, we study the completeness for two examples of such a linear space. The first one is related to an inner product defined by an absolutely continuous matrix measure with respect to a scalar measure such that their entries are rational functions. The second one corresponds to a Sobolev type inner product wit h a unique mass point at 0 in the discrete component involving the derivatives.

Notations. As usual, we denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers, non-negative integers, respectively. We set $\mathbb{C}_N = \mathbb{C}_{1\times N}$. We identify $\mathbb{C}_{1\times 1}$ with $\mathbb{C}$ and $\mathbb{C}^{\geqslant}_{1\times 1}$ with the set of non-negative real numbers. If $A \in \mathbb{C}_{N\times N}$, then A^* stands for its adjoint matrix. If $A \in \mathbb{C}_{N\times N}$ is nonsingular, then A^{-1} will denote its inverse. We set $\mathbb{P}_{v,N} = \mathbb{P}_{1\times N}$. The elements of this set are called vector polynomials. In all Hilbert spaces of functions considered in the paper, $[v]$ will denote the class of equivalence generated by a function v. For a subset S of the complex plane we denote by $\mathfrak{B}(S)$ the set of all Borel subsets of S.

2. Construction of an isometric transformation: density of polynomials

Let f be an arbitrary $\mathbb{C}$-valued function on L'_N. We will deal with the following operator

$$R_{m,N}(f)(\lambda) = \frac{1}{N\lambda^m}\sum_{k=0}^{N-1}\varepsilon^{-mk}f(\lambda\varepsilon^k), \tag{15}$$

where $\varepsilon = \cos\frac{2\pi}{N} + i\sin\frac{2\pi}{N}$. When $f = p$ is a polynomial, this operator was introduced in [2, p. 90]. In such a case, it becomes

$$R_{m,N}(p)(\lambda) = \sum_j a_{jN+m}\lambda^{jN}, \quad p(\lambda) = \sum_i a_i\lambda^i \in \mathbb{P}. \tag{16}$$

Notice that the following relation holds

$$R_{m,N}(f)(\lambda\varepsilon^k) = R_{m,N}(f)(\lambda), \quad k = 0, 1, \ldots, N-1. \tag{17}$$

Indeed, we can write

$$\begin{aligned} R_{m,N}(f)(\lambda\varepsilon) &= \frac{1}{N\lambda^m\varepsilon^m}\sum_{k=0}^{N-1}\varepsilon^{-mk}f(\lambda\varepsilon^{k+1}) = \frac{1}{N\lambda^m\varepsilon^m}\sum_{j=1}^{N}\varepsilon^{-m(j-1)}f(\lambda\varepsilon^j)\\ &= \frac{1}{N\lambda^m}\sum_{j=1}^{N}\varepsilon^{-mj}f(\lambda\varepsilon^j) = \frac{1}{N\lambda^m}\left[\sum_{j=1}^{N-1}\varepsilon^{-mj}f(\lambda\varepsilon^j) + f(\lambda)\right] = R_{m,N}(f)(\lambda), \end{aligned}$$

and, therefore, (17) follows.
On the other hand,

$$\sum_{m=0}^{N-1}\lambda^m R_{m,N}(f)(\lambda) = f(\lambda), \quad \lambda \in L'_N. \tag{18}$$

Indeed, we have

$$\begin{aligned} \sum_{m=0}^{N-1}\lambda^m R_{m,N}(f)(\lambda) &= \frac{1}{N}\sum_{m=0}^{N-1}\sum_{k=0}^{N-1}\varepsilon^{-mk}f(\lambda\varepsilon^k) = \sum_{k=0}^{N-1}\left(\frac{1}{N}\sum_{m=0}^{N-1}\varepsilon^{-mk}\right)f(\lambda\varepsilon^k)\\ &= \sum_{k=0}^{N-1}\delta_{k,0}f(\lambda\varepsilon^k) = f(\lambda). \end{aligned}$$

We will denote

$$\vec{f}_r(\lambda) = (R_{0,N}(f)(\lambda), R_{1,N}(f)(\lambda), \ldots, R_{N-1,N}(f)(\lambda)). \tag{19}$$

Notice that

$$\vec{f}_s(\lambda)J_\lambda = \vec{f}_r(\lambda). \tag{20}$$

Hence, the inner product in $L^2_{\mathbf{M},A}$ can be written as follows:

$$\begin{aligned}(f,g)_{\mathbf{M},A} &= \int_{L'_N} \vec{f}_r(\lambda)\mathbf{M}'_\tau(\lambda)\vec{g}_r^*(\lambda)d\tau_{\mathbf{M}} + \vec{f}_d A\vec{g}_d^* \\ &= \int_{L'_N} \vec{f}_r(\lambda)\mathbf{M}'_\tau(\lambda)\vec{g}_r^*(\lambda)d\tau_{\mathbf{M}} + \vec{f}_t\widehat{A}\vec{g}_t^*, \quad f,g \in L^2_{\mathbf{M},A}, \end{aligned} \tag{21}$$

where

$$\widehat{A} = (\widehat{a}_{i,j})_{i,j=0}^{N-1} = \mathrm{diag}(0!, 1!, \ldots, (N-1)!)A\,\mathrm{diag}(0!, 1!, \ldots, (N-1)!).$$

As usual, $\mathrm{diag}(l_0, l_1, \ldots, l_{N-1})$ denotes the $N \times N$ diagonal matrix with entries l_j in the main diagonal.

Consider the space $L^2_{\mathbf{M},A}$. Let f, g be functions in $L^2_{\mathbf{M},A}$ and set

$$\Psi(f,g;\lambda) = \vec{f}_r(\lambda)\mathbf{M}'_\tau(\lambda)\vec{g}_r^*(\lambda). \tag{22}$$

Thus we can split the inner product in two integrals as follows:

$$\begin{aligned}(f,g)_{\mathbf{M},A} &= \int_{L'_N} \Psi(f,g;\lambda)d\tau_{\mathbf{M}} + \vec{f}_t\widehat{A}\vec{g}_t^* \\ &= \sum_{j=0}^{N-1}\int_{L'_{N,2j}} \Psi(f,g;\lambda)d\tau_{\mathbf{M}} && (23) \\ &\quad + \sum_{j=0}^{N-1}\int_{L'_{N,2j+1}} \Psi(f,g;\lambda)d\tau_{\mathbf{M}} && (24) \\ &\quad + \vec{f}_t\widehat{A}\vec{g}_t^*. && (25)\end{aligned}$$

If k is an even (resp. odd) non-negative integer number, then the mapping

$$x = \varphi_k(\lambda) = \lambda^N$$

is a bijection from $L'_{N,k}$ onto $\mathbb{R}_+ = (0, +\infty)$ (resp. $\mathbb{R}_- = (-\infty, 0)$). Its inverse mapping

$$\lambda = \psi_k(x) = \sqrt[N]{x},$$

with the adequate choice of a branch of the root, is a continuous function.
For $\Delta \in \mathfrak{B}(L'_{N,k})$, we denote

$$\varphi_k(\Delta) := \{x : x = \varphi_k(\lambda), \ \lambda \in \Delta\}, \quad k = 0, 1, \ldots, 2N-1.$$

For $\Delta' \in \mathfrak{B}(\mathbb{R}_+)$, we denote $\psi_k(\Delta') := \{\lambda : \lambda = \psi_k(x), \ x \in \Delta'\}$ where k is an even non-negative number less than $2N$.
For $\Delta'' \in \mathfrak{B}(\mathbb{R}_-)$, we denote $\psi_k(\Delta'') := \{\lambda : \lambda = \psi_k(x), \ x \in \Delta''\}$ where k is an odd non-negative number less than $2N$.
If k is an even non-negative number less than $2N$, then we define

$$B(\mathbb{R}_+) := \{\widetilde{\Delta} \subseteq \mathbb{R}_+ : \ \widetilde{\Delta} = \varphi_k(\Delta), \ \Delta \in \mathfrak{B}(L'_{N,k})\}. \tag{26}$$

If k is an odd non-negative number less than $2N$, then we define

$$B(\mathbb{R}_-) := \{\widetilde{\Delta} \subseteq \mathbb{R}_- : \ \widetilde{\Delta} = \varphi_k(\Delta), \ \Delta \in \mathfrak{B}(L'_{N,k})\}. \tag{27}$$

Since φ_k is bijective and continuous,

$$B(\mathbb{R}_+) \supseteq \mathfrak{B}(\mathbb{R}_+), \quad B(\mathbb{R}_-) \supseteq \mathfrak{B}(\mathbb{R}_-).$$

In fact, $B(\mathbb{R}_\pm)$ are σ-algebras containing closed intervals. As a straightforward consequence of the minimality of the σ-algebras of the Borel subsets we deduce that

$$B(\mathbb{R}_\pm) = \mathfrak{B}(\mathbb{R}_\pm).$$

Thus, we can define positive measures $\widehat{\tau}_{\mathbf{M},k}$ on $\mathfrak{B}(\mathbb{R}_+)$ in the following way.
If k is an even non-negative number less than $2N$, then

$$\widehat{\tau}_{\mathbf{M},k}(\widetilde{\Delta}) := \tau_{\mathbf{M}}(\psi_k(\widetilde{\Delta})), \quad \widetilde{\Delta} \in \mathfrak{B}(\mathbb{R}_+). \tag{28}$$

In a similar way we can define positive measures $\widehat{\tau}_{\mathbf{M},k}$ on $\mathfrak{B}(\mathbb{R}_-)$ as follows:
If k is an odd non-negative number less than $2N$, then

$$\widehat{\tau}_{\mathbf{M},k}(\widetilde{\Delta}) := \tau_{\mathbf{M}}(\psi_k(\widetilde{\Delta})), \quad \widetilde{\Delta} \in \mathfrak{B}(\mathbb{R}_-). \tag{29}$$

Using the change of variable in (23) and (24), we obtain

$$(f, g)_{\mathbf{M},A} = \sum_{j=0}^{N-1} \int_{\mathbb{R}_+} \Psi(f, g; \psi_{2j}(x)) d\widehat{\tau}_{\mathbf{M},2j} \tag{30}$$

$$+ \sum_{j=0}^{N-1} \int_{\mathbb{R}_-} \Psi(f, g; \psi_{2j+1}(x)) d\widehat{\tau}_{\mathbf{M},2j+1} + \vec{f}_t \widehat{A} \vec{g}_t^{\,*}. \tag{31}$$

Indeed, let k be an even non-negative number less than $2N$. Let us consider

$$I_k := \int_{L'_{N,k}} \Psi(f, g; \lambda) d\tau_{\mathbf{M}} = \lim_{n\to+\infty} \int_{L'_{N,k}} \Psi_n(f, g; \lambda) d\tau_{\mathbf{M}},$$

where $\{\Psi_n(f, g; \lambda)\}_{n=1}^{\infty}$ is a sequence of simple functions uniformly convergent to $\Psi(f, g; \lambda)$ on $L'_{N,k}$ Notice that the existence of such a sequence is a consequence of the definition of Lebesgue integral for bounded non-negative measures (see [9]).

Let $\{y_{n,j}\}_{j=1}^{\infty}$ be all pairwise different values of $\Psi_n(f, g; \lambda)$. If

$$L'_{N,k}(n, j) = \{\lambda \in L'_{N,k} : \ \Psi_n(f, g; \lambda) = y_{n,j}\}, \tag{32}$$

then we get

$$\varphi_k(L'_{N,k}(n, j)) = \{x \in \mathbb{R}_+ : \ \Psi_n(f, g; \psi_k(x)) = y_{n,j}\}.$$

Thus

$$I_k = \lim_{n\to+\infty} \sum_{j=1}^{\infty} y_{n,j} \tau_{\mathbf{M}}(L'_{N,k}(n, j)) = \lim_{n\to+\infty} \sum_{j=1}^{\infty} y_{n,j} \widehat{\tau}_{\mathbf{M},k}(\varphi_k(L'_{N,k}(n, j)))$$

$$= \lim_{n\to+\infty} \int_{\mathbb{R}_+} \Psi_n(f, g; \psi_k(x)) d\widehat{\tau}_{\mathbf{M},k}.$$

As a composition of measurable functions, $\Psi(f, g;\, \psi_k(x))$ is $\mathfrak{B}(\mathbb{R}_+)$-measurable. Simple functions $\Psi_n(f, g;\, \psi_k(x))$ converge uniformly on $\mathbb{R}_+$ to $\Psi(f, g;\, \psi_k(x))$, and then the limit

$$\lim_{n\to+\infty} \int_{\mathbb{R}_+} \Psi_n(f, g;\, \psi_k(x)) d\widehat{\tau}_{\mathbf{M},k} = I_k$$

exists. Thus,

$$I_k = \int_{\mathbb{R}_+} \Psi(f, g;\, \psi_k(x)) d\widehat{\tau}_{\mathbf{M},k}. \tag{33}$$

The case of an odd k can be analyzed in a similar way.
Set

$$\widehat{\mathbf{M}}_k(\Delta') = \int_{\Delta'} \mathbf{M}'_\tau(\psi_k(x)) d\widehat{\tau}_{\mathbf{M},k}, \tag{34}$$

where $\Delta' \in \mathfrak{B}(\mathbb{R}_+)$, k is even, $0 \leqslant k \leqslant 2N-1$. Let us write

$$\widehat{\mathbf{M}}_k(\Delta'') = \int_{\Delta''} \mathbf{M}'_\tau(\psi_k(x)) d\widehat{\tau}_{\mathbf{M},k}, \tag{35}$$

where $\Delta'' \in \mathfrak{B}(\mathbb{R}_-)$, k is odd, $0 \leqslant k \leqslant 2N-1$.
Taking into account that $\widehat{\mathbf{M}}_k = (\widehat{m}_{k;i,j})_{i,j=0}^{N-1}$, $0 \leqslant k \leqslant 2N-1$, are finite positive semi-definite matrix measures and using the definition of the integral with respect to a matrix of measures (see [15]), we can write

$$(f, g)_{\mathbf{M},A} = \sum_{j=0}^{N-1} \int_{\mathbb{R}_+} \vec{f}_r(\psi_{2j}(x)) d\widehat{\mathbf{M}}_{2j} \vec{g}_r^*(\psi_{2j}(x)) \tag{36}$$

$$+ \sum_{j=0}^{N-1} \int_{\mathbb{R}_-} \vec{f}_r(\psi_{2j+1}(x)) d\widehat{\mathbf{M}}_{2j+1} \vec{g}_r^*(\psi_{2j+1}(x)) + \vec{f}_t \widehat{A} \vec{g}_t^*. \tag{37}$$

Consider the following scalar measure:

$$\sigma_+(\Delta_+) = \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} \widehat{m}_{2j;i,i}(\Delta_+), \quad \Delta_+ \in \mathfrak{B}(\mathbb{R}_+). \tag{38}$$

We will denote by $(\widehat{\mathbf{M}}_k)'_{\sigma_+}$ the Radon–Nikodym derivative of the matrix measure $\widehat{\mathbf{M}}_k$ with respect to σ_+. In a similar way, we set

$$\sigma_-(\Delta_-) = \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} \widehat{m}_{2j+1;i,i}(\Delta_-), \quad \Delta_- \in \mathfrak{B}(\mathbb{R}_-), \tag{39}$$

where $(\widehat{\mathbf{M}}_k)'_{\sigma_-}$ is the Radon–Nikodym derivative of the matrix measure $\widehat{\mathbf{M}}_k$ with respect to σ_-. Then

$$(f, g)_{\mathbf{M},A} = \sum_{j=0}^{N-1} \int_{\mathbb{R}_+} \vec{f}_r(\psi_{2j}(x)) (\widehat{\mathbf{M}}_{2j})'_{\sigma_+} \vec{g}_r^*(\psi_{2j}(x)) d\sigma_+ \tag{40}$$

$$+ \sum_{j=0}^{N-1} \int_{\mathbb{R}_-} \vec{f}_r(\psi_{2j+1}(x)) (\widehat{\mathbf{M}}_{2j+1})'_{\sigma_-} \vec{g}_r^*(\psi_{2j+1}(x)) d\sigma_- + \vec{f}_t \widehat{A} \vec{g}_t^*. \tag{41}$$

(17) shows that $\vec{f}_r(\psi_k(x))$ and $\vec{g}_r(\psi_k(x))$ do not depend on k (that is on the choice of a branch of the root). Thus, we can write

$$(f,g)_{\mathbf{M},A} = \int_{\mathbb{R}_+} \vec{f}_r(\sqrt[N]{x}) \left(\sum_{j=0}^{N-1} (\widehat{\mathbf{M}}_{2j})'_{\sigma_+} \right) \vec{g}_r^*(\sqrt[N]{x}) d\sigma_+ \tag{42}$$

$$+ \int_{\mathbb{R}_-} \vec{f}_r(\sqrt[N]{x}) \left(\sum_{j=0}^{N-1} (\widehat{\mathbf{M}}_{2j+1})'_{\sigma_-} \right) \vec{g}_r^*(\sqrt[N]{x}) d\sigma_- + \vec{f}_t \widehat{A} \vec{g}_t^*. \tag{43}$$

Let $\Delta \in \mathfrak{B}(\mathbb{R})$. It has a unique decomposition

$$\Delta = \Delta_- \cup \Delta_0 \cup \Delta_+, \tag{44}$$

where $\Delta_\pm \in \mathfrak{B}(\mathbb{R}_\pm)$, $\Delta_0 \in \mathfrak{B}(\{0\})$ with $\Delta_\pm = \Delta \cap \mathbb{R}_\pm$, $\Delta_0 = \Delta \cap \{0\}$.

Let us consider the following scalar measure

$$\sigma(\Delta) = \sigma_-(\Delta_-) + \sigma_0(\Delta_0) + \sigma_+(\Delta_+), \quad \Delta \in \mathfrak{B}(\mathbb{R}), \tag{45}$$

where

$$\sigma_0(\Delta_0) = \begin{cases} 0, & \Delta_0 = \emptyset, \\ 1, & \Delta_0 = \{0\}. \end{cases} \tag{46}$$

Define the following $\mathbb{C}_{N\times N}$-valued function on $\mathbb{R}$

$$\mathbf{D}(x) = \begin{cases} \sum_{j=0}^{N-1} (\widehat{\mathbf{M}}_{2j})'_{\sigma_+}, & x \in \mathbb{R}_+, \\ \sum_{j=0}^{N-1} (\widehat{\mathbf{M}}_{2j+1})'_{\sigma_-}, & x \in \mathbb{R}_-, \\ \widehat{A}, & x = 0. \end{cases} \tag{47}$$

Then

$$(f,g)_{\mathbf{M},A} = \int_{\mathbb{R}_- \cup \mathbb{R}_+} \vec{f}_r(\sqrt[N]{x}) \mathbf{D}(x) \vec{g}_r^*(\sqrt[N]{x}) d\sigma + \vec{f}_t \mathbf{D}(0) \vec{g}_t^*. \tag{48}$$

On the other hand, we introduce the following mapping:

$$(\mathbf{V}f)(x) = \begin{cases} \vec{f}_r(\sqrt[N]{x}), & x \in \mathbb{R}_- \cup \mathbb{R}_+, \\ \vec{f}_t, & x = 0, \end{cases} \quad f \in L^2_{\mathbf{M},A}. \tag{49}$$

Using this definition we can rewrite (48) as follows:

$$(f,g)_{\mathbf{M},A} = \int_{\mathbb{R}} (\mathbf{V}f)(x) \mathbf{D}(x) (\mathbf{V}g)(x) d\sigma. \tag{50}$$

Set

$$\widetilde{\mathbf{M}}(\Delta) = \int_\Delta \mathbf{D} d\sigma, \quad \Delta \in \mathfrak{B}(\mathbb{R}). \tag{51}$$

taking into account that $\widetilde{\mathbf{M}} = (\widetilde{m}_{i,j})_{i,j=0}^{N-1}$ is a finite positive semi-definite matrix measure on $\mathfrak{B}(\mathbb{R})$, we can represent our inner product as follows:

$$(f, g)_{\mathbf{M},A} = \int_{\mathbb{R}} (\mathbf{V}f)(x) d\widetilde{\mathbf{M}}(\mathbf{V}g)(x). \tag{52}$$

Denote by $L_{2,\widetilde{\mathbf{M}}}$ the space of (classes of equivalence of) $\mathbb{C}_{1\times N}$-valued functions on $\mathbb{R}$ which are square integrable with respect to the matrix measure $\widetilde{\mathbf{M}}$ [15]. Let $(\cdot,\cdot)_{\widetilde{\mathbf{M}}}$, $\|\cdot\|_{\widetilde{\mathbf{M}}}$ be the inner product and the corresponding norm in $L_{2,\widetilde{\mathbf{M}}}$, respectively. Then we can write

$$(f, g)_{\mathbf{M},A} = (\mathbf{V}f, \mathbf{V}g)_{\widetilde{\mathbf{M}}}, \quad f, g \in L^2_{\mathbf{M},A}. \tag{53}$$

From the above identity, $\mathbf{V}$ is an isometric transformation from $L^2_{\mathbf{M},A}$ onto $L_{2,\widetilde{\mathbf{M}}}$ in such a way that it maps classes of equivalence in $L^2_{\mathbf{M},A}$ into classes of equivalence in $L_{2,\widetilde{\mathbf{M}}}$. The range of $\mathbf{V}$ is a subset $S_{2,\widetilde{\mathbf{M}}} \subseteq L_{2,\widetilde{\mathbf{M}}}$. Let us describe $S_{2,\widetilde{\mathbf{M}}}$.

Proposition 2.1. *The set $S_{2,\widetilde{\mathbf{M}}}$ consists of those classes of equivalence $[\cdot]$ in $L_{2,\widetilde{\mathbf{M}}}$ which include at least one representative $v(x) = (v_0(x), v_1(x), \ldots, v_{N-1}(x))$ with finite values such that the function*

$$u_v = u_v(\lambda) = \begin{cases} \sum_{m=0}^{N-1} \lambda^m v_m(\lambda^N), & \lambda \in L'_N, \\ v_0(0), & \lambda = 0, \end{cases} \tag{54}$$

has a finite $\|\cdot\|_{\mathbf{M},A}$ norm, and if $A \neq 0$ then

$$\frac{u_v^{(j)}(0)}{j!} = v_j(0), \quad 1 \leqslant j \leqslant \mathfrak{r}(A). \tag{55}$$

Remark. In the above proposition we need to deal with representatives of classes since we do not know a priori that the transformation (54) maps classes onto classes.

Proof. Denote the set of classes of equivalence by S. Let us first show that $S \subseteq S_{2,\widetilde{\mathbf{M}}}$. Let $[v] \in S$ such that $v(x) = (v_0(x), v_1(x), \ldots, v_{N-1}(x))$. Given the function u_v defined in (54), we shall find $\mathbf{V}u_v$. Notice that

$$\begin{aligned} R_{m,N}(u_v)(\lambda) &= \frac{1}{N\lambda^m} \sum_{k=0}^{N-1} \varepsilon^{-mk} u_v(\lambda\varepsilon^k) = \frac{1}{N\lambda^m} \sum_{k=0}^{N-1} \varepsilon^{-mk} \sum_{d=0}^{N-1} \lambda^d \varepsilon^{kd} v_d(\lambda^N) \\ &= \frac{1}{N\lambda^m} \sum_{d=0}^{N-1} \lambda^d v_d(\lambda^N) \sum_{k=0}^{N-1} \varepsilon^{(d-m)k} \\ &= \frac{1}{N\lambda^m} \lambda^m v_m(\lambda^N) N = v_m(\lambda^N), \quad \lambda \in L'_N. \end{aligned} \tag{56}$$

From (49) we can write

$$(\mathbf{V}u_v)(x) = \begin{cases} v(x), & x \in \mathbb{R}_- \cup \mathbb{R}_+, \\ \left(v_0(0), \frac{u_v'(0)}{1!}, \frac{u_v''(0)}{2!}, \ldots, \frac{u_v^{(N-1)}(0)}{(N-1)!}\right), & x = 0. \end{cases} \tag{57}$$

Here, as before, we will write zeros in those places where derivatives do not exist. From (55) and (57) we get that $\mathbf{V}[u_v] = [v]$. As a consequence, $S \subseteq S_{2,\widetilde{\mathbf{M}}}$.

Next, we will prove that $S_{2,\widetilde{\mathbf{M}}} \subseteq S$. Let consider an arbitrary class $[v] \in S_{2,\widetilde{\mathbf{M}}}$. There exists a function f with finite values and a finite $\|\cdot\|_{\mathbf{M},A}$ norm such that $\mathbf{V}f = v$, where v is a representative of the class $[v]$ with finite values. This means that

$$v_m(x) = R_{m,N}(f)(\sqrt[N]{x}), \quad x \in \mathbb{R}\backslash\{0\},\ m = 0, 1, \ldots, 2N-1, \tag{58}$$

and if $A \neq 0$, then

$$v_j(0) = \frac{f^{(j)}(0)}{j!}, \quad 0 \leqslant j \leqslant \mathfrak{r}(A). \tag{59}$$

If $A = 0$, then we can set $f(0) = v_0(0)$. Thus, in any case we have

$$f(0) = v_0(0). \tag{60}$$

From (58) and (17) it follows that

$$R_{m,N}(f)(\lambda) = v_m(\lambda^N), \quad \lambda \in L'_N,\ m = 0, 1, \ldots, 2N-1. \tag{61}$$

Given the representative v we define the function u_v as in (54). Thus, the relation (56) holds for u_v. From (61) and (56) we get

$$R_{m,N}(f)(\lambda) = R_{m,N}(u_v)(\lambda), \quad \lambda \in L'_N,\ m = 0, 1, \ldots, 2N-1. \tag{62}$$

By (18) we obtain that

$$f(\lambda) = u_v(\lambda), \quad \lambda \in L'_N. \tag{63}$$

Furthermore, from (54) and (60) we get

$$f(\lambda) = u_v(\lambda), \quad \lambda \in L_N. \tag{64}$$

Thus, the function u_v has a finite $\|\cdot\|_{\mathbf{M},A}$ norm and (59) yields (55). As a consequence, $S_{2,\widetilde{\mathbf{M}}} \subseteq S$. $\square$

Let denote by $C_{2,\widetilde{\mathbf{M}}}$ a set of (classes of equivalence which include) continuous $\mathbb{C}_{1\times N}$-valued functions from $L_{2,\widetilde{\mathbf{M}}}$. It is well known that $C_{2,\widetilde{\mathbf{M}}}$ is a dense subset of $L_{2,\widetilde{\mathbf{M}}}$ [6]. $T_{2,\widetilde{\mathbf{M}}}$ will denote a set of (classes of equivalence which include) continuous $\mathbb{C}_{1\times N}$-valued functions v from $L_{2,\widetilde{\mathbf{M}}}$ which are vector polynomials in $\{x \in \mathbb{R} :\ |x| < \varepsilon_v\}$, where $\varepsilon_v > 0$ depends on v.

Proposition 2.2. *Every function in $C_{2,\widetilde{\mathbf{M}}}$ can be approximated by functions in $T_{2,\widetilde{\mathbf{M}}}$.*

Proof. First, $T_{2,\widetilde{\mathbf{M}}}$ is a subset of $C_{2,\widetilde{\mathbf{M}}}$.
We denote by $R_{2,\widetilde{\mathbf{M}}}$ a set of (classes of equivalence which include) $\mathbb{C}_{1\times N}$-valued functions v from $L_{2,\widetilde{\mathbf{M}}}$ such that:

(i) v is constant in $\{x \in \mathbb{R} : |x| < \varepsilon_v., , , , , \}$, where $\varepsilon_v > 0$ depends on v.
(ii) v is continuous in intervals $(-\infty, -\varepsilon_v]$ and $[\varepsilon_v, +\infty)$.

Notice that a function $v \in R_{2,\widetilde{\mathbf{M}}}$ can have jumps at points $\pm\varepsilon_v$.
Next, we choose an arbitrary function $v = (v_0, v_1, \ldots, v_{N-1}) \in C_{2,\widetilde{\mathbf{M}}}$ as well as an arbitrary real number δ with $0 < \delta < 1$ and consider the following function $v_{(\delta)} = (v_{\delta;0}, v_{\delta;1}, \ldots, v_{\delta;N-1})$

$$v_{(\delta)} = v_{(\delta)}(x) = \begin{cases} v(0), & |x| < \delta, \\ v(x), & |x| \geqslant \delta. \end{cases} \tag{65}$$

Taking into account that $v_{(\delta)}(x) \in R_{2,\widetilde{\mathbf{M}}}$ we get

$$\|v - v_{(\delta)}\|^2_{\widetilde{\mathbf{M}}} = \int_{0<|x|<\delta} (v(x) - v_{(\delta)}(x))\widetilde{\mathbf{M}}'_{\mu}(v(x) - v_{(\delta)}(x))^* d\mu_{\widetilde{\mathbf{M}}},$$

where $\widetilde{\mathbf{M}}'_{\mu} = (\widetilde{m}'_{\mu;i,j})_{i,j=0}^{N-1}$ denotes the Radon–Nikodym derivative of $\widetilde{\mathbf{M}} = (\widetilde{m}_{i,j})_{i,j=0}^{N-1}$ with respect to the trace measure $\mu_{\widetilde{\mathbf{M}}} = \sum_{i=0}^{N-1} \widetilde{m}_{i,i}$. Since all functions under the integral are bounded, the measure $\mu_{\widetilde{\mathbf{M}}}$ is finite, and $|\widetilde{m}'_{\mu;i,j}| \leqslant 1$, $\mu_{\widetilde{\mathbf{M}}}$-a.e., we can write

$$\begin{aligned} \|v - v_{(\delta)}\|^2_{\widetilde{\mathbf{M}}} &= \sum_{i,j=0}^{N-1} \int_{0<|x|<\delta} (v_i(x) - v_i(0))\widetilde{m}'_{\mu;i,j}\overline{(v_j(x) - v_j(0))} d\mu_{\widetilde{\mathbf{M}}} \\ &\leqslant \sum_{i,j=0}^{N-1} C_1^2 \int_{0<|x|<\delta} d\mu_{\widetilde{\mathbf{M}}} = N^2 C_1^2 \mu_{\widetilde{\mathbf{M}}}(0 < |x| < \delta), \end{aligned} \tag{66}$$

where

$$C_1 = \max_{0 \leqslant j \leqslant N-1;\ x \in [-1,1]} |v_j(x) - v_j(0)|.$$

Relation (66) shows that we can approximate v by functions $v_{(\delta)}$.

On the other hand, let $v \in R_{2,\widetilde{\mathbf{M}}}$ which is constant in $(-\varepsilon_v, \varepsilon_v)$ for $\varepsilon_v > 0$. Choose a real number ν with $0 < \nu < 1$, and consider the following function $v_{[\nu]} = (v_{[\nu];0}, v_{[\nu];1}, \ldots, v_{[\nu];N-1})$:

$$v_{[\nu]} = v_{[\nu]}(x) = \begin{cases} v(x), & |x| < \varepsilon_v, \\ v(x), & |x| > \varepsilon_v + \nu, \\ \frac{(|x|-\varepsilon_v)v((\varepsilon_v+\nu)sgn(x)) - (|x|-\varepsilon_v-\nu)v(0)}{\nu}, & \varepsilon_v \leqslant |x| \leqslant \varepsilon_v + \nu, \end{cases} \tag{67}$$

where

$$sgn(x) = \begin{cases} 1, & x \geqslant 0, \\ -1, & x < 0. \end{cases} \tag{68}$$

$v_{[\nu]}(x)$ is continuous and belongs to $T_{2,\widetilde{\mathbf{M}}}$. For $B = \{x : \ \varepsilon_v \leqslant |x| \leqslant \varepsilon_v + \nu\}$ we have

$$\begin{aligned} |v_{[\nu];j}(x)| &\leqslant \left| \frac{|x| - \varepsilon_v}{\nu} \right| |v_j((\varepsilon_v + \nu)sgn(x))| + \left| \frac{|x| - \varepsilon_v - \nu}{\nu} \right| |v_j(0)| \\ &\leqslant |v_j((\varepsilon_v + \nu)sgn(x))| + |v_j(0)| \leqslant 2C_2, \end{aligned} \tag{69}$$

where

$$C_2 = \max_{0 \leqslant j \leqslant N-1;\ x \in [-\varepsilon_v - 1, -\varepsilon_v] \cup [\varepsilon_v, \varepsilon_v + 1]} |v_j(x)|.$$

We can write

$$\begin{aligned}\|v - v_{[\nu]}\|^2_{\widetilde{\mathbf{M}}} &= \int_B (v(x) - v_{[\nu]}(x))\widetilde{\mathbf{M}}'_\mu (v(x) - v_{[\nu]}(x))^* d\mu_{\widetilde{\mathbf{M}}} \\ &= \sum_{i,j=0}^{N-1} \int_B (v_i(x) - v_{[\nu];i}(x))\widetilde{m}'_{\mu;i,j}\overline{(v_j(x) - v_{[\nu];j}(x))}d\mu_{\widetilde{\mathbf{M}}} \\ &\leqslant \sum_{i,j=0}^{N-1} C_3^2 \int_B d\mu_{\widetilde{\mathbf{M}}} = N^2 C_3^2 \mu_{\widetilde{\mathbf{M}}}(B), \end{aligned} \tag{70}$$

where

$$C_3 = \max_{0\leqslant j\leqslant N-1;\ x\in[-\varepsilon_\nu-1,-\varepsilon_\nu]\cup[\varepsilon_\nu,\varepsilon_\nu+1]} |v_j(x)| + C_2.$$

Relation (70) shows that the function v can be approximated by functions $v_{[\nu]}$. Thus, every function in $C_{2,\widetilde{\mathbf{M}}}$ can be approximated by functions of $T_{2,\widetilde{\mathbf{M}}}$. □

As an immediate consequence, we get

Corollary 2.1. *$T_{2,\widetilde{\mathbf{M}}}$ is a dense subset of $L_{2,\widetilde{\mathbf{M}}}$.*

Let $T_{2,\widetilde{\mathbf{M}};0}$ be a subset of $L_{2,\widetilde{\mathbf{M}}}$ which consists of (classes of equivalence which include) functions f such that

$$f(x) = g(x)\chi_{[-K,K]}(x), \quad g(x) \in T_{2,\widetilde{\mathbf{M}}},\ K > 0, \tag{71}$$

where $\chi_{[-K,K]}$ is the characteristic function of the interval $[-K, K]$.

Corollary 2.2. *The set $T_{2,\widetilde{\mathbf{M}};0}$ is dense in $L_{2,\widetilde{\mathbf{M}}}$.*

Proof. From the σ-additivity of the Lebesgue integral it follows that the function from $T_{2,\widetilde{\mathbf{M}}}$ can be approximated by functions belonging to $T_{2,\widetilde{\mathbf{M}};0}$. □

On the other hand, the following inclusion hold:

Proposition 2.3.

$$T_{2,\widetilde{\mathbf{M}};0} \subseteq S_{2,\widetilde{\mathbf{M}}}. \tag{72}$$

Proof. Choose an arbitrary (representative of the class) $v = (v_0, v_1, \dots, v_{N-1}) \in T_{2,\widetilde{\mathbf{M}};0}$ like (71) with $g \in T_{2,\widetilde{\mathbf{M}}}$ and $K > 0$. Notice that it is bounded, continuous in $[-K, K]$, and supported on $[-K, K]$. Therefore the function

$$u_v = u_v(\lambda) = \begin{cases} \sum_{m=0}^{N-1} \lambda^m v_m(\lambda^N), & \lambda \in L'_N, \\ v_0(0), & \lambda = 0, \end{cases}$$

is also bounded, continuous in $S_K = \{z \in L_N : z^N \in [-K, K]\}$, and its support is contained in S_K. This function is a polynomial in a neighborhood of zero. So, it has all derivatives at zero. The function

$$\vec{u}_{vr}(\lambda) = (R_{0,N}(u_v)(\lambda), R_{1,N}(u_v)(\lambda), \dots, R_{N-1,N}(u_v)(\lambda))$$

is continuous on $S_K\backslash\{0\}$ and its support is contained in S_K. Let us show that it is bounded at zero. Choose a non-negative integer number k such that $0 \leqslant k \leqslant 2N - 1$. From l'Hôpital's rule we get

$$\begin{aligned}\lim_{z\in L_{N,k};\ z\to 0} R_{m,N}(u_v)(z) &= \lim_{x\to +0} R_{m,N}(u_v)(x\widehat{\varepsilon}^k)\\ &= \frac{1}{N}\lim_{x\to+0}\frac{\sum_{j=0}^{N-1}\varepsilon^{-mj}u_v(x\widehat{\varepsilon}^k\varepsilon^j)}{(x\widehat{\varepsilon}^k)^m} = \frac{1}{N}\lim_{x\to+0}\frac{\sum_{j=0}^{N-1}\varepsilon^{-mj}u_v'(x\widehat{\varepsilon}^k\varepsilon^j)\varepsilon^j}{m(x\widehat{\varepsilon}^k)^{m-1}}\\ &= \cdots = \frac{1}{N}\lim_{x\to+0}\frac{\sum_{j=0}^{N-1}u_v^{(m)}(x\widehat{\varepsilon}^k\varepsilon^j)}{m!} = \frac{u_v^{(m)}(0)}{m!}.\end{aligned}$$

As a consequence, $\vec{u}_{vr}(\lambda)$ is bounded. Therefore, u_v has a finite $L^2_{\mathbf{M},A}$ norm. Let us check that relation (55) holds. For $0 \leqslant d \leqslant N-1$ Leibniz's formula yields

$$\begin{aligned}u_v^{(d)}(0) &= \sum_{m=0}^{N-1}(\lambda^m v_m(\lambda^N))^{(d)}(0) = \sum_{m=0}^{N-1}\sum_{l=0}^{d} C_d^l(\lambda^m)^{(l)}(v_m(\lambda^N))^{(d-l)}\Big|_{\lambda=0}\\ &= \sum_{m=0}^{N-1}\sum_{l=0}^{\min(d,m)} C_d^l(\lambda^m)^{(l)}(v_m(\lambda^N))^{(d-l)}\Big|_{\lambda=0},\end{aligned} \tag{73}$$

where $C_d^l = \frac{d!}{l!(d-l)!}$ is the binomial coefficient. Non-zero summands in (73) appear only when $l = m$ if $m \leqslant d$. Therefore

$$u_v^{(d)}(0) = \sum_{m=0}^{d} C_d^m m!(v_m(\lambda^N))^{(d-m)}\Big|_{\lambda=0} = d!v_d(0).$$

According to Proposition 2.1 we obtain $v \in S_{2,\widetilde{\mathbf{M}}}$. $\square$

Corollary 2.3. *The set $S_{2,\widetilde{\mathbf{M}}}$ is dense in $L_{2,\widetilde{\mathbf{M}}}$.*

Proposition 2.4. *The transformation $\mathbf{V}$ maps polynomials which belong to $L^2_{\mathbf{M},A}$ onto $\mathbb{C}_{1\times N}$-valued vector polynomials in $L_{2,\widetilde{\mathbf{M}}}$. The inverse transformation $\mathbf{V}^{-1}$ maps $\mathbb{C}_{1\times N}$-valued vector polynomials which belong to $L_{2,\widetilde{\mathbf{M}}}$ into polynomials in $L^2_{\mathbf{M},A}$.*

Proof. If we use relation (16), then the first statement follows directly from the definition of $\mathbf{V}$.
Let us prove the second statement. Let $v(x) = (v_0(x), v_1(x), \ldots, v_{N-1}(x))$ be in (a class of equivalence in) $L_{2,\widetilde{\mathbf{M}}}$. Notice that $[v] \in T_{2,\widetilde{\mathbf{M}}}$. Consider the following functions:

$$w(K;x) = (w_0(K;x), w_1(K;x), \ldots, w_{N-1}(K;x)) = v(x)\chi_{[-K,K]}(x), \quad K \in \mathbb{N}. \tag{74}$$

As in the proof of the previous proposition, we can associate to them some functions

$$u_{w,K}(\lambda) = \begin{cases}\sum_{m=0}^{N-1}\lambda^m w_m(K;\lambda^N), & \lambda \in L_N'\\ w_0(K;0), & \lambda = 0\end{cases}, \quad K \in \mathbb{N},$$

which have finite $L^2_{\mathbf{M},A}$ norm and

$$u_{w,K}^{(d)}(0) = d!w_d(K;0). \tag{75}$$

Moreover, as in the proof of Proposition 2.1, we can write

$$\mathbf{V}[u_{w,K}] = [w(K;\cdot)], \quad K \in \mathbb{N}. \tag{76}$$

Therefore

$$\|u_{w,K}\|_{\mathbf{M},A} = \|w(K;x)\|_{\widetilde{\mathbf{M}}} \leqslant \|v(x)\|_{\widetilde{\mathbf{M}}}, \quad K \in \mathbb{N}. \tag{77}$$

Set

$$u_v(\lambda) = \begin{cases} \sum_{m=0}^{N-1} \lambda^m v_m(\lambda^N), & \lambda \in L'_N, \\ v_0(0), & \lambda = 0. \end{cases} \tag{78}$$

Notice that

$$u_{w,K}(\lambda) = u_v(\lambda)\chi_{S_K}(\lambda), \tag{79}$$

where $S_K = \{z \in L_N : z^N \in [-K, K]\}$. By (77) the norms $\|u_v(\lambda)\chi_{S_K}(\lambda)\|_{\mathbf{M},A}$ are uniformly bounded. Therefore $u_v \in L^2_{\mathbf{M},A}$. On the other hand, from (75) and (79)

$$u_v^{(d)}(0) = d! v_d(0). \tag{80}$$

Thus, conditions of Proposition 2.1 are satisfied for u_v. In its proof it was shown that

$$\mathbf{V}[u_v] = [v], \quad K \in \mathbb{N}, \tag{81}$$

According to (78), u_v is a polynomial. □

Proposition 2.5. *Complex polynomials which belong to $L^2_{\mathbf{M},A}$ are dense in $L^2_{\mathbf{M},A}$ if and only if $\mathbb{C}_{1\times N}$-valued vector polynomials which belong to $L_{2,\widetilde{\mathbf{M}}}$ are dense in $L_{2,\widetilde{\mathbf{M}}}$.*

Proof. First, let us prove sufficiency. Assume that $\mathbb{C}_{1\times N}$-valued vector polynomials from $L_{2,\widetilde{\mathbf{M}}}$ are dense therein. Let $f \in L^2_{\mathbf{M},A}$. Set $v = \mathbf{V}f \in L_{2,\widetilde{\mathbf{M}}}$. Then there exists a vector polynomial p such that $\|v - p\|_{\widetilde{\mathbf{M}}} < \varepsilon$, $\varepsilon > 0$. Then

$$\|\mathbf{V}^{-1}v - \mathbf{V}^{-1}p\|_{\widetilde{\mathbf{M}}} < \varepsilon,$$

and, from Proposition 2.4, $\mathbf{V}^{-1}p$ is a complex polynomial.

Let us check necessity. Assume that complex polynomials which belong to $L^2_{\mathbf{M},A}$ are dense therein. Let $v \in L_{2,\widetilde{\mathbf{M}}}$. According to Corollary 2.3, v can be approximated by $\mathbf{V}f$, where $f \in L^2_{\mathbf{M},A}$, i.e.

$$\|v - \mathbf{V}f\|_{\widetilde{\mathbf{M}}} < \frac{\varepsilon}{2}, \quad \varepsilon > 0. \tag{82}$$

On the other hand, one can approximate f by a complex polynomial r

$$\|f - r\|_{\mathbf{M},A} = \|\mathbf{V}f - \mathbf{V}r\|_{\widetilde{\mathbf{M}}} < \frac{\varepsilon}{2}. \tag{83}$$

Then

$$\|v - \mathbf{V}r\|_{\widetilde{\mathbf{M}}} < \varepsilon.$$

But, according to Proposition 2.4, $\mathbf{V}r$ is a vector polynomial. Thus the result follows. □

Denote by $L_{2,\widetilde{\mathbf{M}},N}$ the space of (classes of equivalence of) $\mathbb{C}_{N\times N}$-valued functions on $\mathbb{R}$ which are square integrable with respect to the matrix measure $\widetilde{\mathbf{M}}$ [15]. Let $(\cdot,\cdot)_{\widetilde{\mathbf{M}},N}$, $\|\cdot\|_{\widetilde{\mathbf{M}},N}$ be the inner product and the norm in $L_{2,\widetilde{\mathbf{M}},N}$, respectively.
Notice that a $\mathbb{C}_{N\times N}$-valued $\mathfrak{B}(\mathbb{R})$-measurable function $F(x)=(f_{i,j}(x))_{i,j=0}^{N-1}$ on $\mathbb{R}$ has a finite $\|\cdot\|_{\widetilde{\mathbf{M}},N}$ norm if and only if its rows

$$F_i=(f_{i,j}(x))_{j=0}^{N-1},\quad i=0,1,\ldots,N-1,$$

are $\mathfrak{B}(\mathbb{R})$-measurable functions with finite $\|\cdot\|_{\widetilde{\mathbf{M}}}$ norms. Moreover, we have

$$\|F\|^2_{\widetilde{\mathbf{M}},A}=\sum_{i=0}^{N-1}\|F_i\|^2_{\widetilde{\mathbf{M}}}. \tag{84}$$

Thus, the linear space $L_{2,\widetilde{\mathbf{M}},N}$ can be considered as a direct sum of N copies of the linear space $L_{2,\widetilde{\mathbf{M}}}$. Relation (84) implies that $\mathbb{C}_{N\times N}$-valued polynomials from $L_{2,\widetilde{\mathbf{M}},N}$ are dense in $L_{2,\widetilde{\mathbf{M}},N}$ if and only if $\mathbb{C}_{1\times N}$-valued polynomials from $L_{2,\widetilde{\mathbf{M}}}$ are dense in $L_{2,\widetilde{\mathbf{M}}}$. Using Proposition 2.5 we get the following result:

Theorem 2.1. *Complex polynomials which belong to $L^2_{\mathbf{M},A}$ are dense in $L^2_{\mathbf{M},A}$ if and only if $\mathbb{C}_{N\times N}$-valued polynomials which belong to $L_{2,\widetilde{\mathbf{M}},N}$ are dense in $L_{2,\widetilde{\mathbf{M}},N}$.*

Now we can apply some results by Lopez-Rodriguez on the density of matrix polynomials to study the density of polynomials in $L^2_{\mathbf{M},A}(L_N)$.

Theorem 2.2. *Let $A\in\mathbb{C}^{\geqslant}_{N\times N}$ and $\mathbf{M}$ be a bounded $\mathbb{C}^{\geqslant}_{N\times N}$-valued measure on $\mathfrak{B}(L'_N)$. Consider a linear space $L^2_{\mathbf{M},A}(L_N)$ and suppose that it includes all complex polynomials. Let us construct the corresponding matrix measure $\widetilde{\mathbf{M}}$ on $\mathfrak{B}(\mathbb{R})$ and assume that the linear space $L^2_{\widetilde{\mathbf{M}},N}$ contains all $\mathbb{C}_{N\times N}$-valued polynomials. If (12) holds for the measure $\widetilde{\mathbf{M}}$ and the corresponding matrix Hamburger moment problem is completely indeterminate, then complex polynomials are dense in $L^2_{\mathbf{M},A}$ if and only if*

$$\int_{\mathbb{R}}\frac{d\widetilde{\mathbf{M}}(x)}{x-\lambda}=-\{C^*(\lambda)[I+U]-iA^*(\lambda)[I-U]\}\{D^*(\lambda)[I+U]$$
$$-iB^*(\lambda)[I-U]\}^{-1},\quad \lambda\in\mathbb{C}\backslash\mathbb{R}, \tag{85}$$

where U is a constant unitary matrix and $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $D(\cdot)$ are holomorphic matrix functions which are computed explicitly using the moments $\{S_k\}_{k=0}^{\infty}$ of the measure $\widetilde{\mathbf{M}}$.

3. The completeness of $L^2_{\mathbf{M},A}(L_N)$

Let us study completeness of the linear space $L^2_{\mathbf{M},A}(L_N)$. If $A\neq 0$ and $\mathfrak{r}(A)>0$, then for an arbitrary function v in $S_{2;\widetilde{\mathbf{M}}}$ and according to Proposition 2.1, the function u_v should have first derivative at zero. This property is not true for all functions in $L_{2;\widetilde{\mathbf{M}}}$. In fact, we can choose $v(x)=(v_0(x),v_1(x),\ldots,v_{N-1}(x))$ with

$$v_j(x)=0,\quad j\geqslant 1,\quad v_0(x)=\chi_{[0,1]}(x).$$

Then the function u_v is discontinuous at zero. Therefore, $S_{2;\widetilde{\mathbf{M}}}$ is dense in $L_{2;\widetilde{\mathbf{M}}}$ but $S_{2;\widetilde{\mathbf{M}}}\neq L_{2;\widetilde{\mathbf{M}}}$. Using the isometric transformation $\mathbf{V}$ we see that $L^2_{\mathbf{M},A}(L_N)$ is not complete.

Consider the case when either $A = 0$ or $A \neq 0$, $\mathfrak{r}(A) = 0$. Choose an arbitrary $[v] \in L_{2;\widetilde{\mathbf{M}}}$ with a representative having finite values $v(x) = (v_0(x), v_1(x), \ldots, v_{N-1}(x))$. Set

$$u_v = u_v(\lambda) = \sum_{m=0}^{N-1} \lambda^m v_m(\lambda^N), \quad \lambda \in L_N. \tag{86}$$

As in the proof of Proposition 2.1 we get

$$R_{m,N}(u_v)(\lambda) = v_m(\lambda^N), \quad \lambda \in L'_N. \tag{87}$$

By (17) we can write

$$R_{m,N}(u_v)(\sqrt[N]{x}) = v_m(x), \quad x \in \mathbb{R}\backslash\{0\}. \tag{88}$$

Notice that

$$R_{0,N}(u_v)(0) = u_v(0) = v_0(0). \tag{89}$$

Using (48)

$$(v, v)_{\widetilde{\mathbf{M}}} = \int_{\mathbb{R}_- \cup \mathbb{R}_+} \vec{u}_{vr}(\sqrt[N]{x})\mathbf{D}(x)\vec{u}_{vr}^{\,*}(\sqrt[N]{x})d\sigma + (\vec{u_v})_t \mathbf{D}(0)(\vec{u_v})_t^{*} \tag{90}$$

and a backward iteration of the previous relations to (48) yields

$$(u_v, u_v)_{\mathbf{M},A} = (v, v)_{\widetilde{\mathbf{M}}} < \infty. \tag{91}$$

Thus, the function u_v has a finite $\| \cdot \|_{\mathbf{M},A}$ norm and from Proposition 2.1 and (89) we conclude that $[v] \in S_{2,\widetilde{\mathbf{M}}}$. Therefore $L_{2,\widetilde{\mathbf{M}}} = S_{2,\widetilde{\mathbf{M}}}$. As a conclusion, we can state

Theorem 3.1. *Let $A \in \mathbb{C}^{\geqslant}_{N\times N}$ and $\mathbf{M}$ be a bounded $\mathbb{C}^{\geqslant}_{N\times N}$-valued measure on $\mathfrak{B}(L'_N)$. The space $L^2_{\mathbf{M},A}(L_N)$ is complete if and only if one of the following conditions hold:*

(i) $A = 0$.
(ii) $A \neq 0$, $\mathfrak{r}(A) = 0$.

Unfortunately, the proof of Theorem 2.1 about the completeness of polynomials in [7, p. 163] was not incorrect and, in a general situation, the statement is not true at all.

4. Examples

We shall consider some examples of spaces $L^2_{\mathbf{M},A}(L_N)$.

1. Let $\tau = \tau(\Delta)$ be a non-negative scalar measure on $\mathfrak{B}(L_N)$. Consider the space L^2_τ of (classes of equivalence of) complex-valued $\mathfrak{B}(L_N)$-measurable functions f defined in L_N, such that

$$\|f\|^2_\tau = \int_{L_N} |f(\lambda)|^2 d\tau < \infty. \tag{92}$$

Let us introduce the following $\mathbb{C}_{N\times N}$-valued non-negative measure on L'_N

$$\mathbf{M}(\Delta) = \int_\Delta J_\lambda^{-1} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} (J_\lambda^{-1})^* d\tau, \quad \Delta \in \mathfrak{B}(L'_N). \tag{93}$$

Notice that the entries of J_λ^{-1} are polynomials in the variable λ. Set

$$A = \begin{pmatrix} \tau(\{0\}) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{94}$$

Relation (9) shows that $f \in L^2_\tau$ if and only if $f \in L^2_{\mathbf{M},A}$. Moreover, the norms of f in these spaces coincide. The inner products will also be equal. Thus, we can use Theorem 2.2 to study the density of polynomials. Theorem 3.1 shows the well known fact that L^2_τ is complete.
2. Let $\tau = \tau(\Delta)$ be a non-negative scalar measure on $\mathfrak{B}(L_N)$. Let $C \in \mathbb{C}_{N\times N}$ such that $C \neq 0$ and $\mathfrak{r}(C) \geqslant 1$. Consider the linear space $W^2_{\tau,C}$ of (classes of equivalence of) complex-valued $\mathfrak{B}(L_N)$-measurable functions f defined in L_N, such that f has $\mathfrak{r}(C)$th derivative at zero and

$$\|f\|^2_{W,\tau,A} = \int_{L_N} |f(\lambda)|^2 d\tau + \left(f(0), f'(0), \dots, f^{(N-1)}(0)\right) C \left(f(0), f'(0), \dots, f^{(N-1)}(0)\right)^* < \infty, \tag{95}$$

where, by convention, we write zeros whenever the derivatives do not exist .
Let introduce the following $\mathbb{C}_{N\times N}$-valued non-negative measure on L'_N.

$$\mathbf{M}(\Delta) = \int_\Delta J_\lambda^{-1} \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} (J_\lambda^{-1})^* d\tau, \quad \Delta \in \mathfrak{B}(L'_N). \tag{96}$$

Set

$$A = \begin{pmatrix} \tau(\{0\}) & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} + C. \tag{97}$$

Relation (9) shows that $f \in W^2_{\tau,C}$ if and only if $f \in L^2_{\mathbf{M},A}$. The corresponding norms and inner products in these spaces coincide. Consequently, we can use Theorem 2.2 to study the density of polynomials. Theorem 3.1 states that $W^2_{\tau,C}$ is not complete.

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